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Suppressing chaos in fractional-order systems by periodic perturbations on system variables

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Abstract. Based on extensive numerical and computer-graphical simulations, it is shown that fractional-order chaotic systems can be stabilized by slightly perturbing the system state variables periodically. In this chaos control scheme, the tunable parameters are chosen empirically. The effectiveness of this chaos control method is demonstrated by fractional-order Lorenz, Chen and Rössler systems, where the underlying initial value problems are numerically integrated by using the Grünwald–Letnikov method.

PACS. 05.45.Ac Low-dimensional chaos – 05.45.Gg Control of chaos, applications of chaos – 05.45.Pq Numerical simulations of chaotic systems – 02.60.Cb Numerical simulation; solution of equations

1 Introduction

Extensive studies have been carried out for chaos control in the last two decades [1–4]. The objective of control is to suppress the chaotic dynamics of a system, so that the system can be driven to some stable states such as a single point or a periodic cycle.

In the study of chaos control, methods based on parameter perturbations are common, motivated by the OGY algorithm [5]. A somewhat less popular method is to apply periodic perturbations on the system state variables [6, 7], primarily conceived for continuous- and discrete-time chaotic systems but was lately extended to a class of discontinuous dynamical systems [8,9]. Even without rigorous mathematical justification, the success of this method has been well demonstrated by both numerical simulations and experimental implementations.

Spurred by the recent research advances in the fractional-order systems, some works on controlling fractional-order systems have been reported. For example, frequency-domain techniques based on Bode diagrams have been suggested, so that one can obtain linear approximations of the fractional integrator (see e.g. [10,11]), thereby chaos control may be accomplished.

However, it is still a challenge to achieve chaos control in fractional-order systems in general. In this paper, it is to verify the possibility of applying periodic perturbations on the system state variables in order to achieve the goal of chaos control, and also to testify its effectiveness on some typical chaotic systems.

To start, consider first a continuous-time autonomous chaotic system modeled by the following ODE Initial Value Problem (IVP)

$$\dot{x} = f(x), \quad x(0) = x_0, \quad t \in I = [0, T],\ T > 0\,, \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function and $x \in \mathbb{R}^n$. The discussions below assume that (1) is of three-dimensional and chaotic, namely $n = 3$, for which many chaotic systems have been reported. Next the system (1) is solved numerically by using some integration method with fixed step size $h > 0$, so it is possible to suppress chaos by perturbing the state variables sporadically at some fixed instants of time. To implement such a chaos control scheme, at every δh time instant, with a positive integer $\delta \in Z^+$, one changes the state variables by

$$x' = x(1 - \gamma), \tag{2}$$

where γ is a relatively small real number ($|\gamma| << 1$). Computationally, this means that x is perturbed when the counted number of integration steps, n, is a multiple of δ, namely $n = m\delta$ with $m \in Z^+$.

The perturbation specified in (2) can be considered as a proportional one since $\frac{x'}{x} = (1-\gamma)$ is a constant. This type of perturbation is referred to as Proportional Perturbation (PP) method hereafter.

Another possible way to perform perturbations may apply additive pulses: $x = x + \gamma$, studied for a class of discontin-

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uous systems in [9], which will not be considered in this paper however.

For illustration, first consider an integer-order Chen system

$$\begin{aligned}\dot{x}_1 &= a(x_2 - x_1),\\ \dot{x}_2 &= x_1(c-a) - x_1x_3 + cx_2,\\ \dot{x}_3 &= x_1x_2 - bx_3\,.\end{aligned}$$

When $a = 35$, $b = 3$ and $c = 28$, it generates a chaotic attractor as shown in Figure 1 (a). It is found that, by perturbing each state variable with $\gamma = -0.005$ in every integration with step size $h = 0.005$ ($\delta = 1$), one can obtain a stable cycle, as plotted in Figure 1 (b).

Here, appropriate values of parameters δ and γ are found only empirically. In [6,7], attempts have been made to find some rules for the selection of δ and γ. Also, as a general remark, special attention has to be paid for determining γ so that the smoothness of the resultant trajectories will not be seriously degraded and the underlying dynamics can remain unchanged structurally.

Therefore, the objective here is to show numerically, with the aid of computer-graphic simulations, that the PP method can be effectively used to provide control for fractional-order chaotic systems. In doing so, the existence of continuous regions for γ is shown, with some fixed values of δ, where chaos control can be guaranteed to achieve.

The organization of the paper is as follows: In Section 2, basic notions about numerical integration of a class of fractional-order IVPs with the Grünwald–Letnikov method are presented. In Section 3 the PP method is applied to three well-known fractional-order chaotic systems, namely the Lorenz, Chen and Rössler systems. Finally, conclusions are drawn in Section 5.

2 Grünwald-Letnikov method for PP algorithm implementation

To implement numerically the PP method we shall use the Grünwald-Letnikov discretization. Generally, fractional-order systems are modeled by the following system of fractional differential equations

$$\frac{d^q}{dt^q}x(t) = f(x(t)), \quad t \in I, \tag{3}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function, $x \in \mathbb{R}^n$, and $q = (q_1, q_2, ..., q_n)^T$ with $q_i \in \mathbb{R}$ represents the fractional order.

As is well known, there are several definitions of fractional derivatives, such as Riemann-Liouville fractional derivative, Caputo fractional derivative, and Grünwald-Letnikov fractional derivative (see, for example, [12–18]). Knowing that fractional derivatives of initial conditions are usually undefined in practical cases, they are thus avoided here. For this reason, Caputo derivative is preferred, since the corresponding initial conditions can be specified in classical forms.

Based on the Caputo operator, (3) can be transformed into the following IVP

$$\begin{aligned}{}^{C}\!D^q_* x(t) = f(x(t)), \quad x^{(k)}(0) = x_0^{(k)},\\ k = 0, 1, 2, ..., \lceil q\rceil - 1,\ t \in I\,,\end{aligned} \tag{4}$$

where $\lceil\cdot\rceil$ denotes the ceiling function that rounds up to the nearest integer and ${}^{C}D^q_*$ represents the Caputo operator of order q. As defined by Caputo in 1969 [12], one has

$${}^{C}\!D^q_* y(t) = \frac{1}{\Gamma(\lceil q\rceil - q)}\int_0^t (t-\tau)^{\lceil q\rceil - q - 1} D^q y(t)d\tau\,,$$

and Γ is the Euler's Gamma function given by

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt, \quad z \in \mathbb{C},\ \Re(z) > 0.$$

For integer q, the definitions of Caputo derivative and classical derivative are identical: ${}^{C}D^q_* y(t) = \frac{d^q}{dt^q}y(t)$. Even though some applications with $q > 1$ have been discussed in recent years, most of the physical phenomena can be modeled with $q \in (0, 1)$. From a qualitatively point of view, these two cases are different (see e.g. [19]). In this paper, only the case of $q \in (0, 1)$ is considered. Accordingly, by taking into account of the Caputo derivative, the initial condition can be transformed to be in the standard form, i.e. $x(0) = x_0$.

For $q \in (0, 1)$, the IVP (4) becomes

$${}^{C}\!D^q_* x(t) = f(x(t)), \quad x(0) = x_0, \quad t \in I\,. \tag{5}$$

Similarly to the integer-order systems, the requirement for the continuity of f ensures the existence of solutions (see page 85 of Chap. 6 in [19] for more details).

There are several numerical methods that can be used to solve the IVP (5), such as the multi-step predictor-corrector Adams-Bashforth-Moulton scheme [20], the Grünwald-Letnikov discretization, etc. Here, the Grünwald-Letnikov discretization (based on finite differences) is adopted, because

1. it is simple, with excellent computational performance;
2. it can handle both commensurate and incommensurate cases without any modification of the coding scheme.

The Caputo derivative can be approximated by the Grünwald-Letnikov derivative. Note that the definitions for both Caputo and Grünwald-Letnikov derivatives are equivalent only in the particular case of autonomous initial conditions, $x_0 = 0$ [18,13], denoted by ${}^{GL}C^q y(t)$ [18] and

$${}^{C}\!D^q_* y(t) \simeq {}^{GL}D^q_t y(t) := \lim_{h\to 0}\frac{1}{h^q}\sum_{j=0}^{\left[\frac{t}{h}\right]} c^q_i y(t - jh)\,, \quad t \in I\,, \tag{6}$$

where the Grünwald-Letnikov coefficients c^q_i are given by

$$c^q_i = (-1)^i \binom{q}{i}, \quad i = 0, 1, 2, \ldots\,.$$

The binomial coefficients $\binom{q}{i}$ can be calculated based on the Gamma function, as

$$\binom{q}{i} = \frac{q!}{(q-i)!} = \frac{\Gamma(q+1)}{\Gamma(i+1)\Gamma(q-i+1)}\,.$$

It is easy to see that $c_0^q = 1$ and c_i^q can be obtained recursively, via

$$c_i^q = \left(1 - \frac{1+q}{q}\right) c_{i-1}^q\,.$$

As a result, the Caputo derivative can be approximated by

$$^C\!D_*^q y(t) \approx \frac{1}{h^q} \sum_{j=0}^{\left[\frac{t}{h}\right]} c_i^q y\,(t - jh)\,, \quad t \in I\,. \tag{7}$$

Once the Caputo derivative approximation (7) is available, one can discretize the IVP (5) on some uniform grid $0 \le t_0 < t_1 < \cdots < t_{N+1} = T$ with $t_{i+1} - t_i = h$. Denote $n+1 = [t/h]$ and consider x_n be the numerical approximation of $x(t)$ at $t = t_n$. Then, the right-hand side of (5) is approximated by $f(x_n)$. By combining (5) and (7), one obtains the following discrete equation

$$\frac{1}{h^q} \sum_{i=0}^{n+1} c_i^q x_{n+1-i} = f(x_n)\,.$$

Hence, an explicit form of x_n based on the Grünwald-Letnikov method can be expressed as

$$x_{n+1} = f(x_n) - \sum_{i=1}^{n+1} c_i^q x_{n+1-i}\,. \tag{8}$$

The accuracy of (8) depends strongly on the number of coefficients c_i^q being used and how accurately they are calculated (for more details on the convergence, see e.g. [21]).

Unlike the operators of integer orders, which imply a finite series, the operators of fractional orders such as the Grünwald-Letnikov operator are non-local and can only be determined by infinite series (see relation (6)). In other words, the next state will depend not only on the current state but also all the states in the past. On one hand, it is useful to model some real systems. On the other hand, it requires much more computational effort. Therefore, a compromising scheme, called "short memory" principle (or "fixed memory" principle) has been proposed. Instead of integrating over the entire interval $[0,t]$, it only considers a period of fixed time interval, $[t-L,t]$ with $L < t \le T$ and some adequate choice of L (see [18,22]). By maintaining relative small increment of errors, this can reduce the computational complexity. For (8), the short memory principle means to increase the lower index ($i = 1$) of the sum with a value corresponding to L [23].

3 Chaos Control in Fractional-Order Systems via PP Method

In this section, using a Matlab implementation of the Grünwald-Letnikov method, it is shown numerically that PP method can be applied to suppress chaos in fractional-order systems. For this purpose, consider three representative chaotic systems:

– Fractional-order Lorenz system [24]

$$\begin{aligned} \tfrac{d^q}{dt^q}x_1 &= \sigma(x_2 - x_1), \\ \tfrac{d^q}{dt^q}x_2 &= x_1\,(r - x_3) - x_2, \\ \tfrac{d^q}{dt^q}x_3 &= x_1x_2 - bx_3\,, \end{aligned}$$

with $q = 0.999$, $\sigma = 10$, $r = 28$ and $b = 8/3$;

– Fractional-order Chen system [25]

$$\begin{aligned} \tfrac{d^q}{dt^q}x_1 &= a(x_2 - x_1), \\ \tfrac{d^q}{dt^q}x_2 &= x_1\,(c - a) - x_1x_3 + cx_2, \\ \tfrac{d^q}{dt^q}x_3 &= x_1x_2 - bx_3\,, \end{aligned}$$

with $q = 0.99$, $a = 35, b = 3$ and $c = 28$;

– Fractional-order Rössler system [26]

$$\begin{aligned} \tfrac{d^q}{dt^q}x_1 &= -x_2 - x_1, \\ \tfrac{d^q}{dt^q}x_2 &= x_1 + ax_2, \\ \tfrac{d^q}{dt^q}x_3 &= b + x_3\,(x_1 - c)\,, \end{aligned}$$

with $q = 0.999$, $a = b = 0.2$ and $c = 5.7$.

A fractional value close to unity is taken for q. This is because it presents the most prominent chaotic behavior as to the integer case which has been well studied. In addition, it is known that chaos is relatively easier to be suppressed if q becomes small. Therefore, a large q (closer to 1) is used in order to verify the proposed approach.

The integration time interval is set to $I = [0, 200]$ and, unless specified, the step size is $h = 0.005$. In order to emphasize the stable cycles, the transients were neglected. Figures 2 (a)–(c) indicate all the possible control domains in the space (γ, δ) for each of the above three chaotic systems, when the PP method is applied. Points $C_1(\gamma_1, \delta_1)$, $C_2(\gamma_1, \delta_1)$ and $C_3(\gamma_1, \delta_1)$ are the representative cases, and the control effects are illustrated by Figures 3, 4 and 5, respectively.

As shown in Figures 2 (a) and (b), a large set of choice for (γ, δ) is possible. The domains look similar, except that δ is larger for the Rössler system. It is noted that the largest admissible value of γ is set to be 0.01, regardless of the choice of δ. This is because, the smoothness of the resultant trajectories will be seriously degraded if γ is above this value, and "chattering" phenomena can be observed. On the contrary, a very different result is obtained in the Chen system. Only a small set of γ, $[0.021, 0.023]$, with $\delta = 1$, can be obtained as shown in Figure 2 (c), for which chaos is suppressed. Since a large value of γ is used, a small h, here $h = 0.002$, is used in order to not affect the smoothness of the trajectories.

4 Discussion

In all the presented cases, the obtained control domains in the space (γ, δ) are composed of at least one horizontal segment, one for each δ. Each segment seems to be continuous along the γ axis. Thus, for an adequately chosen δ^*, one obtains a segment for chaos suppression, MN (Fig. 2 (d)), governed by the values of γ_{min} and γ_{max}, where γ_{min} depends on δ^*, and γ_{max} is specified as shown in Figures 2 (a)–(c). Therefore, for some pair (δ^*, γ^*) in MN, chaos can indeed be suppressed. If the chaos can be suppressed for some pair $(\delta^*, \gamma^*) \in MN$, then it can be suppressed for all pairs pair $(\delta^*, \gamma) \in MN$ with $\gamma \in (\gamma^*, \gamma_{max}]$.
In the above examples, perturbations (2) with the same fixed γ have been applied to all state variables. Next, the approach is further generalized by applying the perturbations only to one or two state variables, or to all state variables with different γ and at different time instants δh. Since only three-dimensional systems are considered, $\delta = (\delta_1, \delta_2, \delta_3)^T$ and $\gamma = (\gamma_1, \gamma_2, \gamma_3)^T$. Thus, (2) becomes

$$x_i' = x_i\,(1-\gamma_i) \text{ at every } \delta_i h \text{ moments with } i = 1, 2, 3.$$

The unperturbed state variable(s) is marked by $*$ (as shown in Figures 6 and 7).
Now, some typical cases are presented. Figure 6 depicts the result when only the first state variable x_1 is perturbed. Using$\gamma_1 = 0.032$ after every $\delta_1 h$, with $\delta_1 = 1$, a stable cycle could be found.
Similarly, considering the Rössler system, if perturbations are applied only to the last two state variables at the same time instants ($\delta_2 = \delta_3 = 10$) with the same $\gamma_2 = \gamma_3 = 0.005$, then a stable cycle can also be obtained, as shown in Figure 7.
Chaos control can also be achieved when different γ and different δ are used. Taking the Lorenz system for illustration, the perturbed system rests at a stable point, as shown in Figure 8 when $\delta = (3, 2, 1)^T$ and $\gamma = (0.0005, 0.001, 0.003)^T$.
Finally, consider the most general case for which all the parameters, namely q, δ and γ, are non-commensurate and the perturbations are applied to all state variables. Again, taking the Lorenz system as an example, it can be verified that chaos is suppressed as seen from Figure 9.
While chaos suppression was relatively easy to implement via the PP method, anti-control of chaos seems to be less attainable. In fact, some cases have been studied preliminarily by considering a relative smaller value of q, say $q = 0.9$. In this case, as compared to the one with larger q (but still smaller than unity), it is relatively difficult to obtain chaotic dynamics. This can be verified by having fewer and smaller regions in the bifurcation diagram where chaos is exhibited.
It is noticed that a few pairs of (γ, δ), as indicated by $A_{1,2,3}$ in Figure 2 (a), can achieve anti-control of chaos as shown in Figures 10 and 11. The parameters used in the Lorenz system for Figure 10 are: $\sigma = 10, r = 10, b = 8/3$ with $\delta = 1$ and $\gamma = -0.01$, while those for Figure 11 are: $\sigma = 10, r = 220, b = 8/3$ with $\delta = 1$ and $\gamma = -0.01$.
Another interesting phenomenon observed is that a stable point can be transformed into a stable cycle via the PP method; for example, as shown in Figure 12, the fractional-order Lorenz system with $\sigma = 10, r = 10, b = 10$, $\delta = 1$ and $\gamma = -0.02$. Also, unlike chaos control where $\gamma > 0$, one must use $\gamma < 0$ here for all cases in anti-control of chaos tested thus far.

5 Conclusions

In this paper it has been shown via computer-graphic simulations that it is possible to suppress chaos in fractional-order dynamical systems, modeled by Grünwald-Letnikov derivative, when one or several state variables are perturbed proportionally. To achieve chaos control, each variable can be periodically perturbed at the same or different time instants by using the derived relation (2).
It is found that the set of all admissible values of γ for chaos suppression consists of some horizontal segments in the plane (γ, δ), one segment for each δ. For chaos suppression, only positive values of γ are found to be useful, while on the other hand, a negative value of γ may help anti-control of chaos. Although the parameters γ and δ are determined empirically, the proposed PP method is effective and easy to implement numerically.
Conceptually, periodic perturbations decrease the state variables in the case of chaos control ($\gamma > 0$). The global energy of the system then arrives at a new level, which forces the system to behave stably. Conversely, by periodically increasing the variables for anti-control of chaos ($\gamma < 0$), the global system energy increases and the system loses its stability gradually.
A possible future investigation alone the same line would be to try aperiodic perturbations. Moreover, applying the technique to fractional-order discontinuous systems with respect to the state variables with $n \geq 3$, or to planar non-autonomous systems with dry friction for example, could also be of real interest.

References

1. G. Chen and X. Dong, *From Chaos to Order: Methodologies, Perspectives and Applications* (World Scientific, Singapore, 1998)
2. G. Chen and X.H. Yu (Eds.), *Chaos Control: Theory and Applications* (Springer, Berlin, 2003)
3. A.L. Fradkov and A.Y. Pogromsky, *Introduction to Control of Oscillations and Chaos* (World Scientific, Singapore, 1999)
4. B.W.K. Ling, H.H.C. Iu and H.K. Lam (Eds.), *Control of Chaos in Nonlinear Circuits and Systems* (World Scientific, Singapore, 2008)
5. E. Ott, C. Grebogi and J.A. Yorke, Phys. Rev. Lett. **64**, 1196 (1990)
6. J. Güémez and M.A. Matías, Phys. Lett. A **181**, 29 (1993)
7. M.A. Matías and J. Güémez, Phys. Rev. Lett/. **72**, 1455 (1994)
8. M.-F. Danca, Chaos, Solitons & Fractals **22**, 605 (2004)

9. M.-F. Danca, Comput. Math. Appl. **64**, 849 (2012)
10. W.M. Ahmada and A.M. Harbb, Chaos, Solitons & Fractals **18**, 693 (2003)
11. A. Oustaloup, F. Levron, B. Mathieu and F.M. Nanot, IEEE Trans. Circuits Syst.–I **47**, 25 (2000)
12. M. Caputo, *Elasticity and Dissipation*, (Zanichelli, Bologna, 1969)
13. L. Dorcak, Numerical Models for the Simulation of the Fractional-Order Control Systems.arXiv:math/0204108v1 (2002)
14. R. Gorenflo and F. Mainardi, Fractional calculus: Integral and differential equations of fractional order, arXiv:0805.3823v1 (2008)
15. K.S. Miller and B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations (John Wiley & Sons, New York, 1993)
16. K.B. Oldham and J. Spanier, *The Fractional Calculus, Theory and Applications of Differentiation and Integration to Arbitrary Order* (Elsevier Science, 1974)
17. A. Oustaloup, *La dérivation non entière: théorie, synthèse et applications* (Hermes, Paris, 1995)
18. I. Podlubny, *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solutiona nd Some of Their Applications* (Academic Press, San Diego, 1999)
19. K. Diethelm and N.J. Ford, J. Math. Anal. Appl. **265**, 229 (2002)
20. K. Diethelm, N.J. Ford and A.D. Freed, Nonlinear Dyn. **29**, 3 (2002)
21. R. Scherer, S.L. Kalla, Y. Tangc and J. Huang, Comput. Math. Appl. **62**, 902 (2011)
22. J.N. Ford and A.C. Simpson, Numerical Algorithms **26**, 333 (2001)
23. I. Petráš, Nonlinear Dyn. **57**, 157 (2009)
24. I. Grigorenko and E. Grigorenko, Phys. Rev. Lett. **91**, 034101 (2003)
25. C.Lia and G.Chen, Chaos, Solitons & Fractals, **22**, 549 (2004)
26. C.Lia, and G. Chen, Phys. A **341**, 55 (2004)

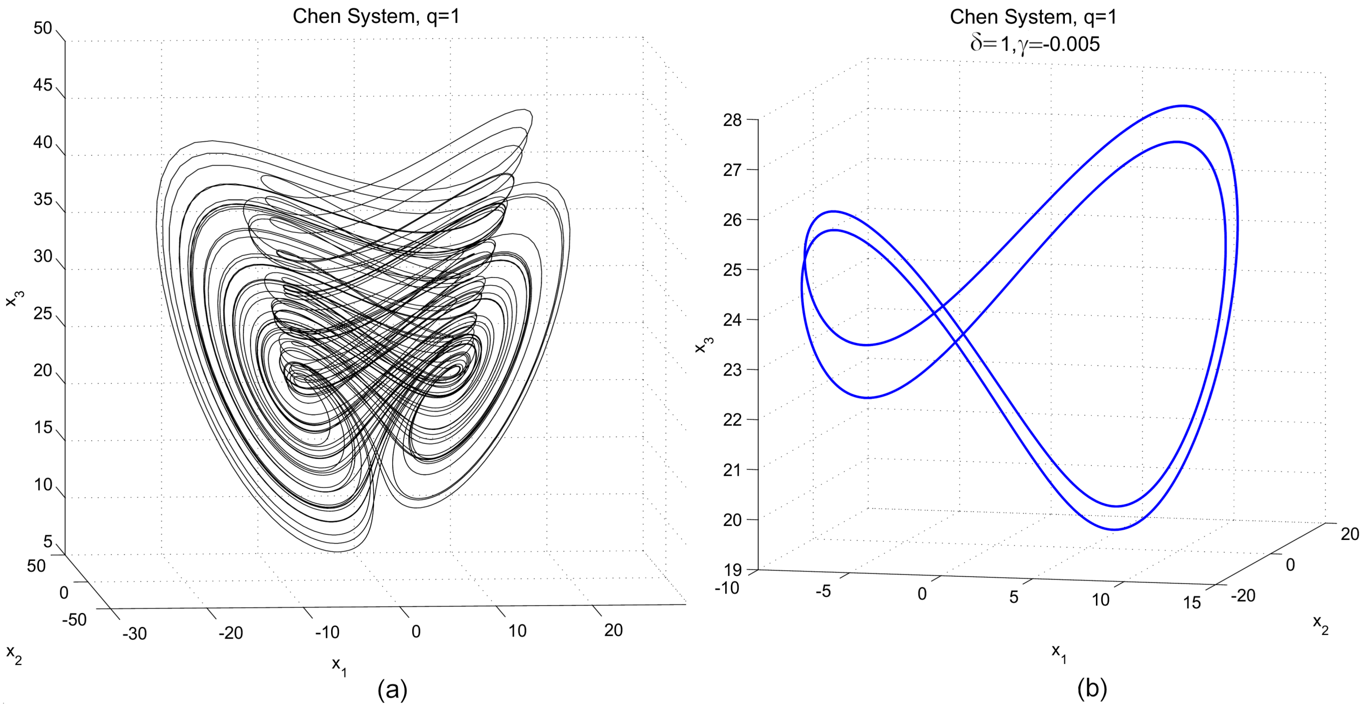


Fig. 1. PP method applied to the integer-order Chen system. $\delta = 1$ and $\gamma = -0.005$.

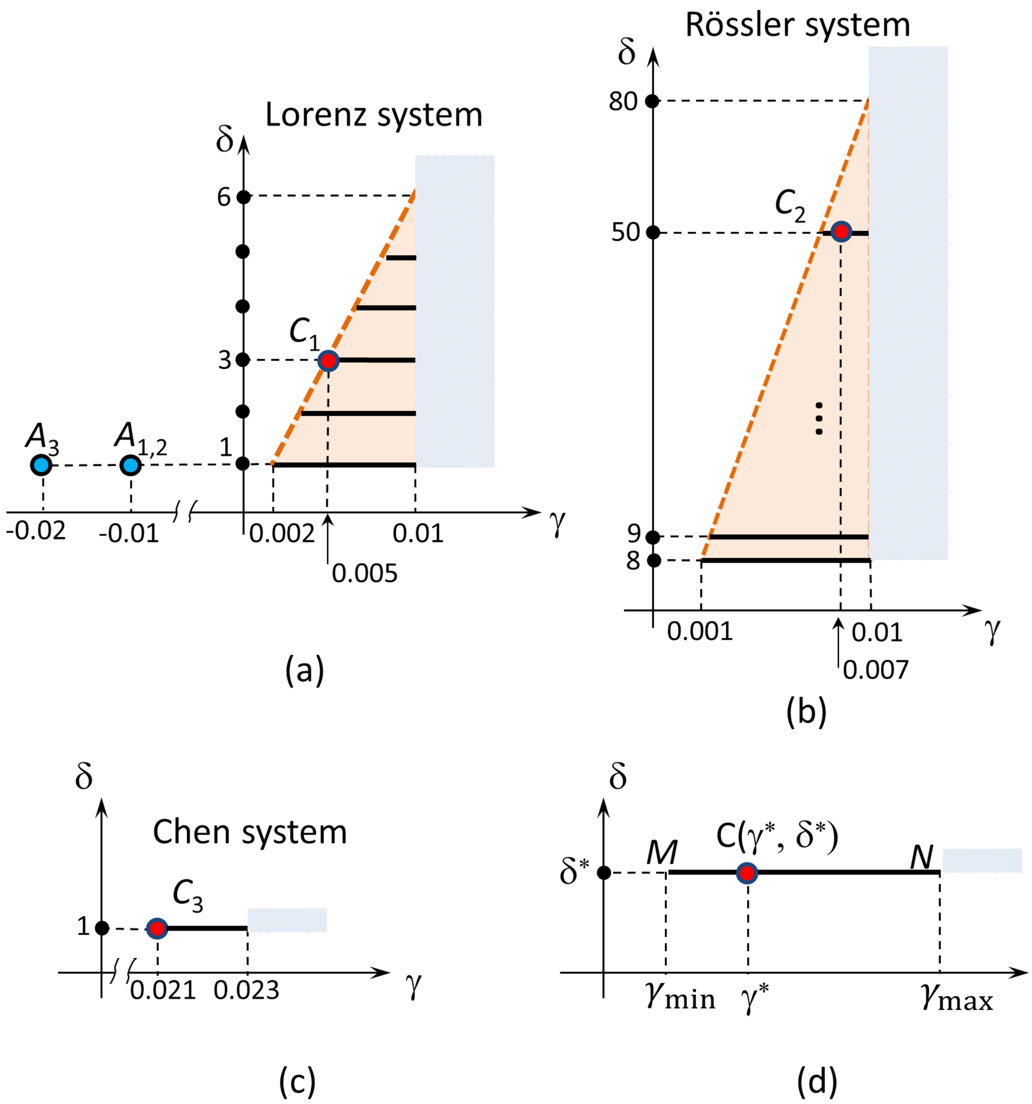


Fig. 2. Sketch for control parameters domains (δ, γ). (a) Lorenz; (b) Rössler; (c) Chen systems; (d) One control segment.

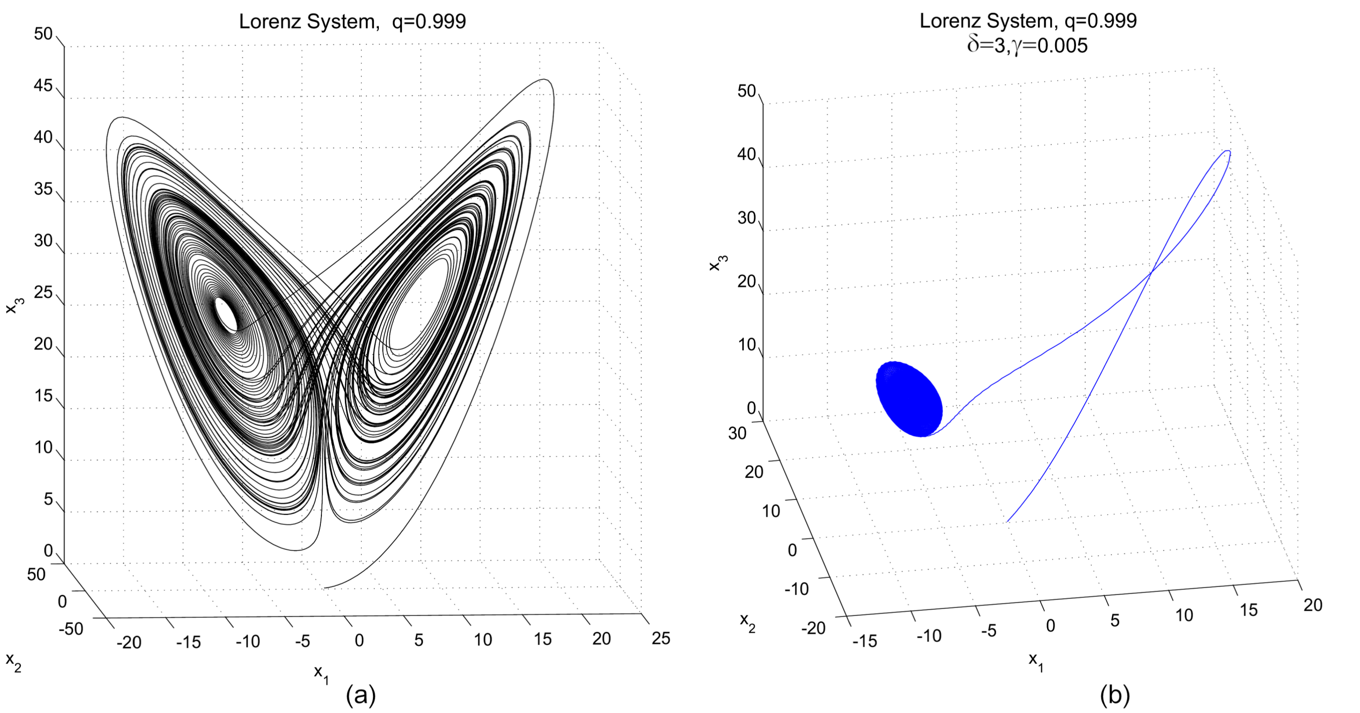


Fig. 3. PP method applied to the Lorenz system with $q = 0.999$. (a) chaotic attractor; (b) stable cycle obtained with $\delta = 3$ and $\gamma = 0.005$.

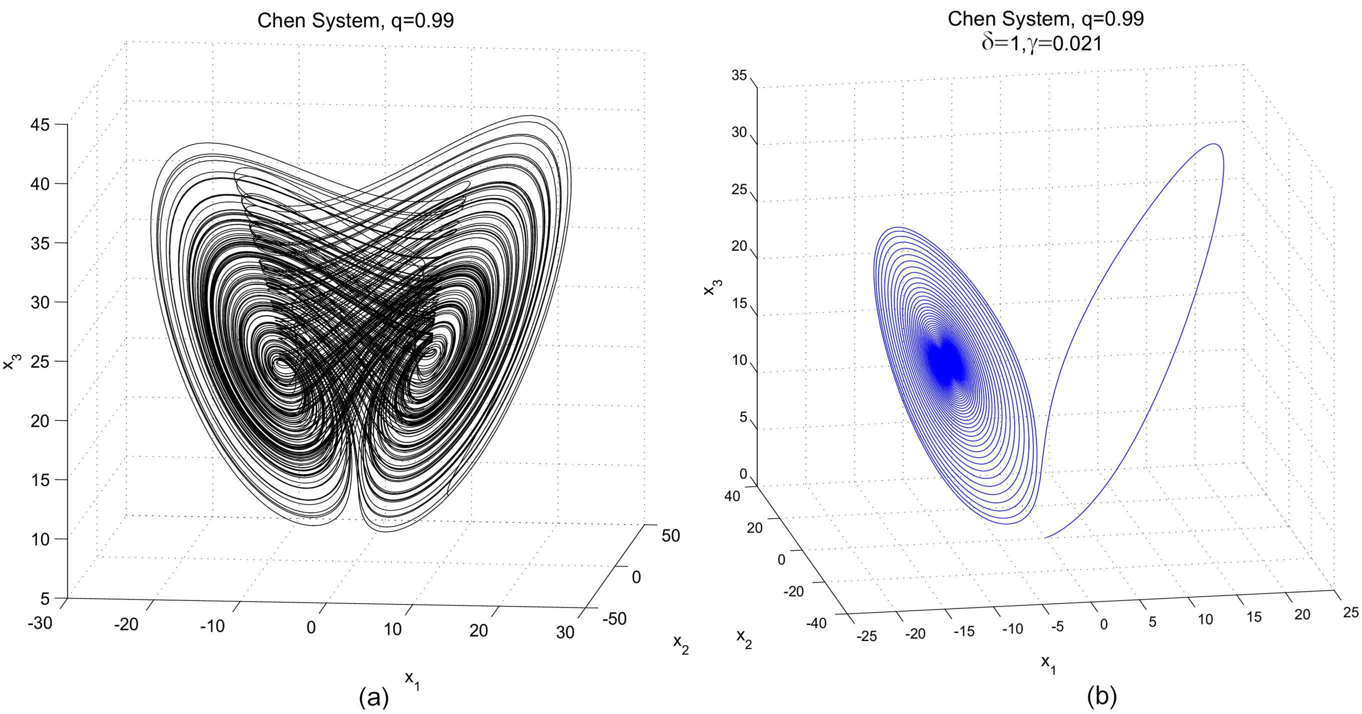


Fig. 4. PP method applied to the Chen system with $q = 0.99$. (a) chaotic attractor; (b) stable point obtained with $\delta = 1$ and $\gamma = 0.021$.

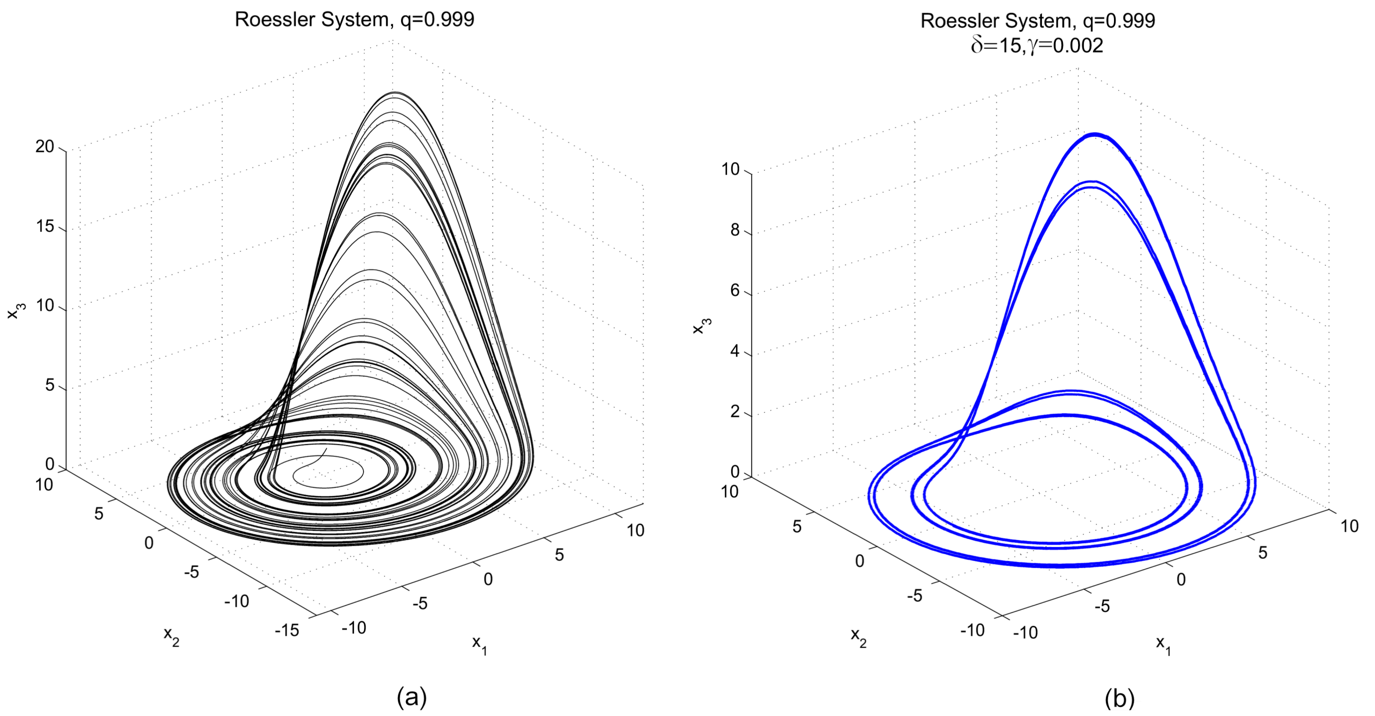


Fig. 5. PP method applied to the Rössler system with $q = 0.999$. (a) chaotic attractor; (b) stable cycle obtained with $\delta = 50$ and $\gamma = 0.007$.

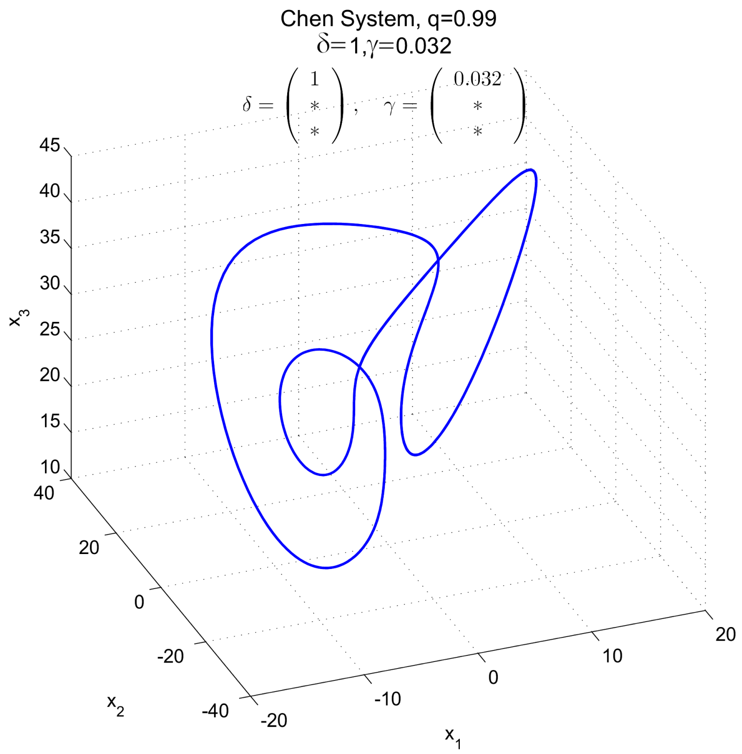


Fig. 6. PP method applied only to x_1 for the Chen system.

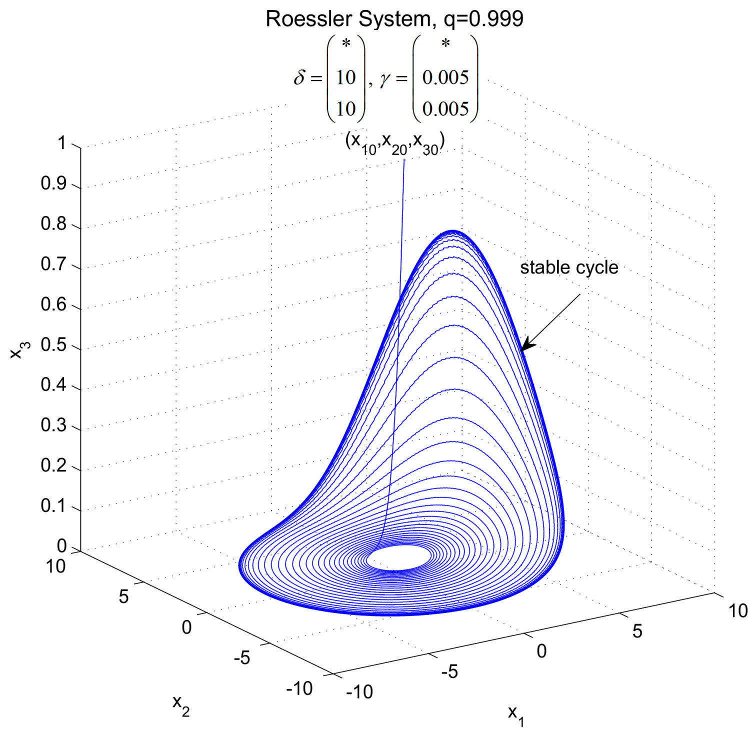


Fig. 7. PP method applied to x_2 and x_3 (with x_1 unperturbed) for the Rössler system.

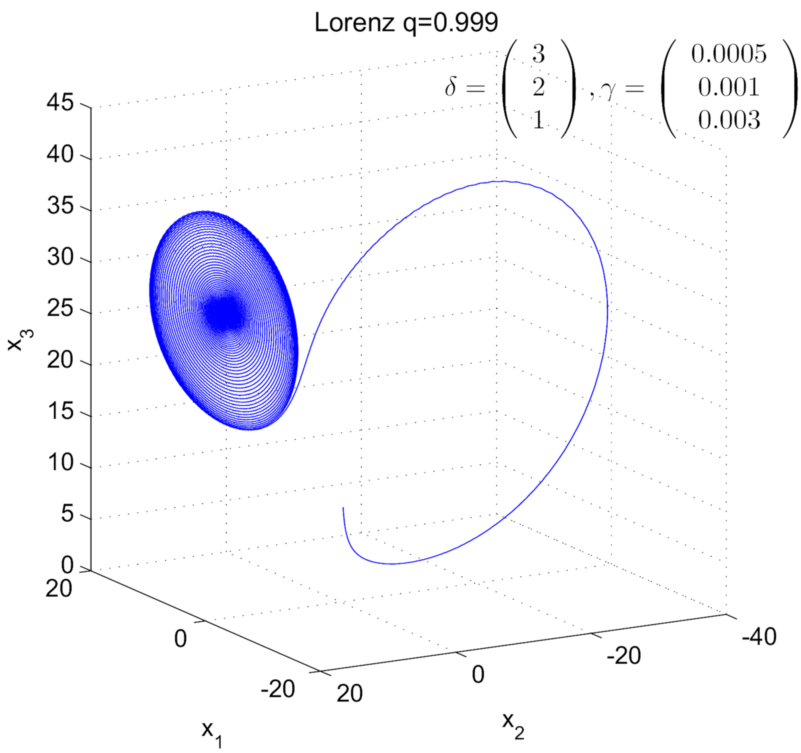


Fig. 8. PP method applied to all variables with different steps δ and different perturbations γ for the Lorenz system.

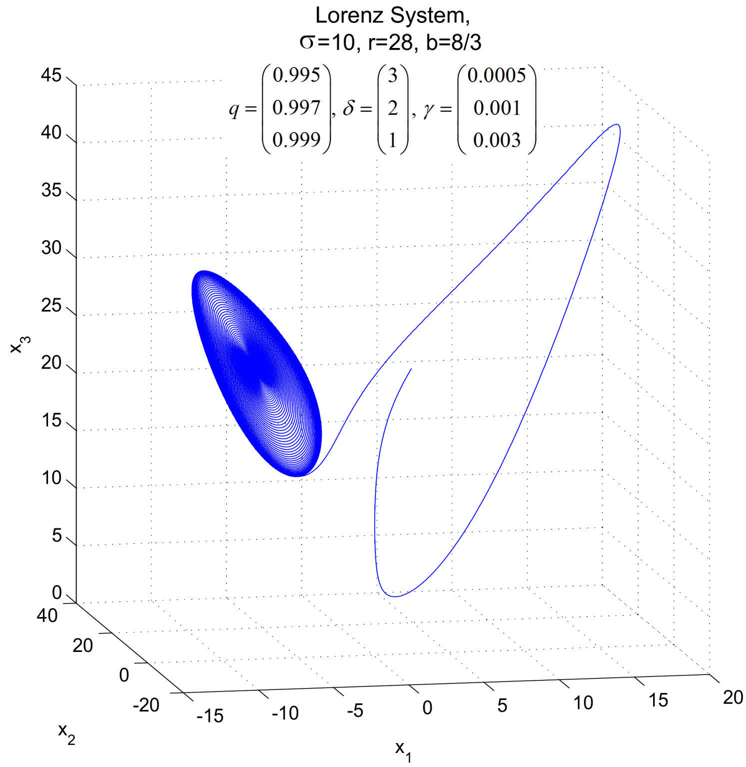


Fig. 9. PP method applied for Lorenz system to all variables with different steps δ and different perturbations γ for the Lorenz system with an incommensurate fractional order.

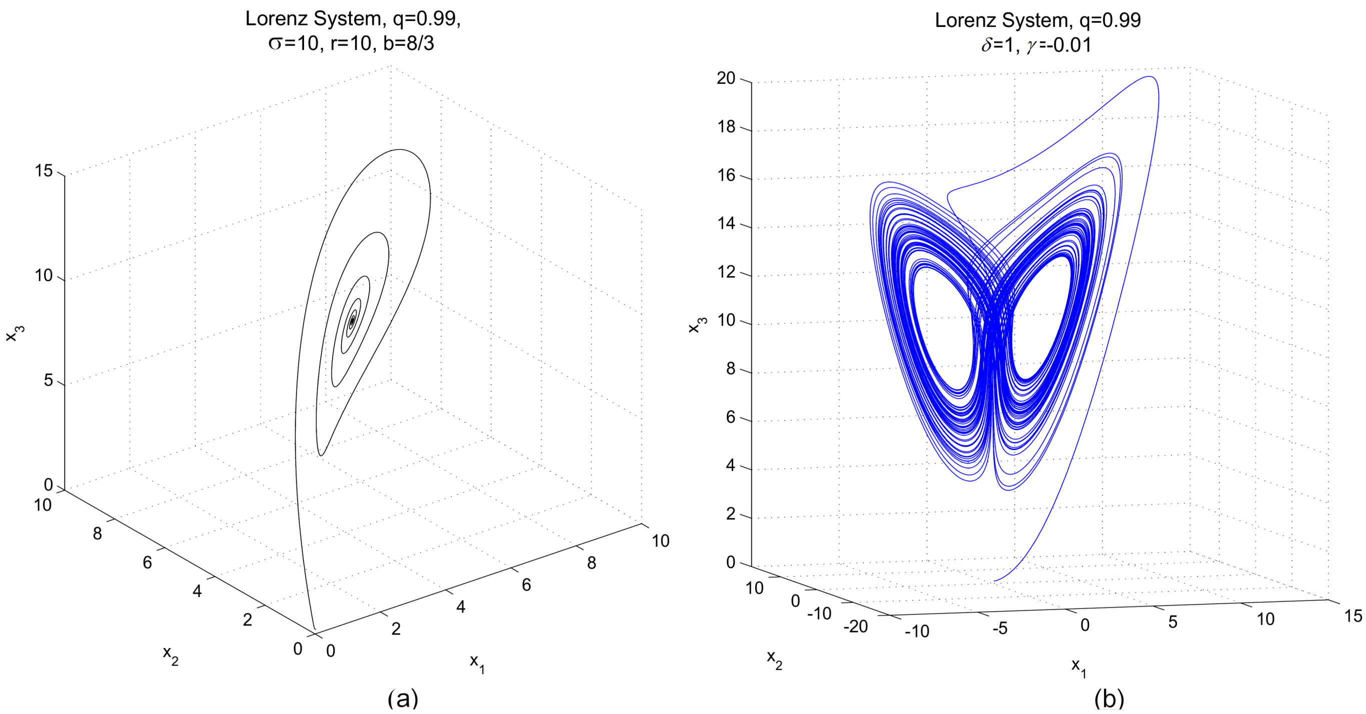
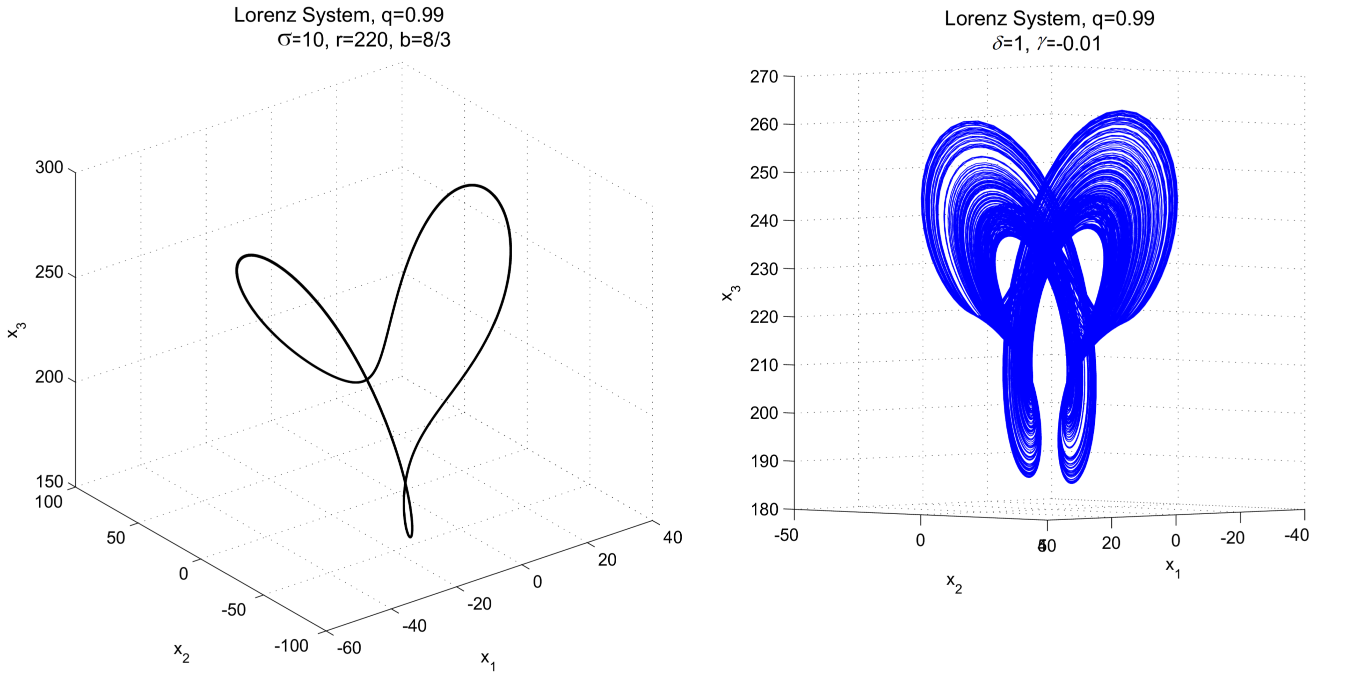
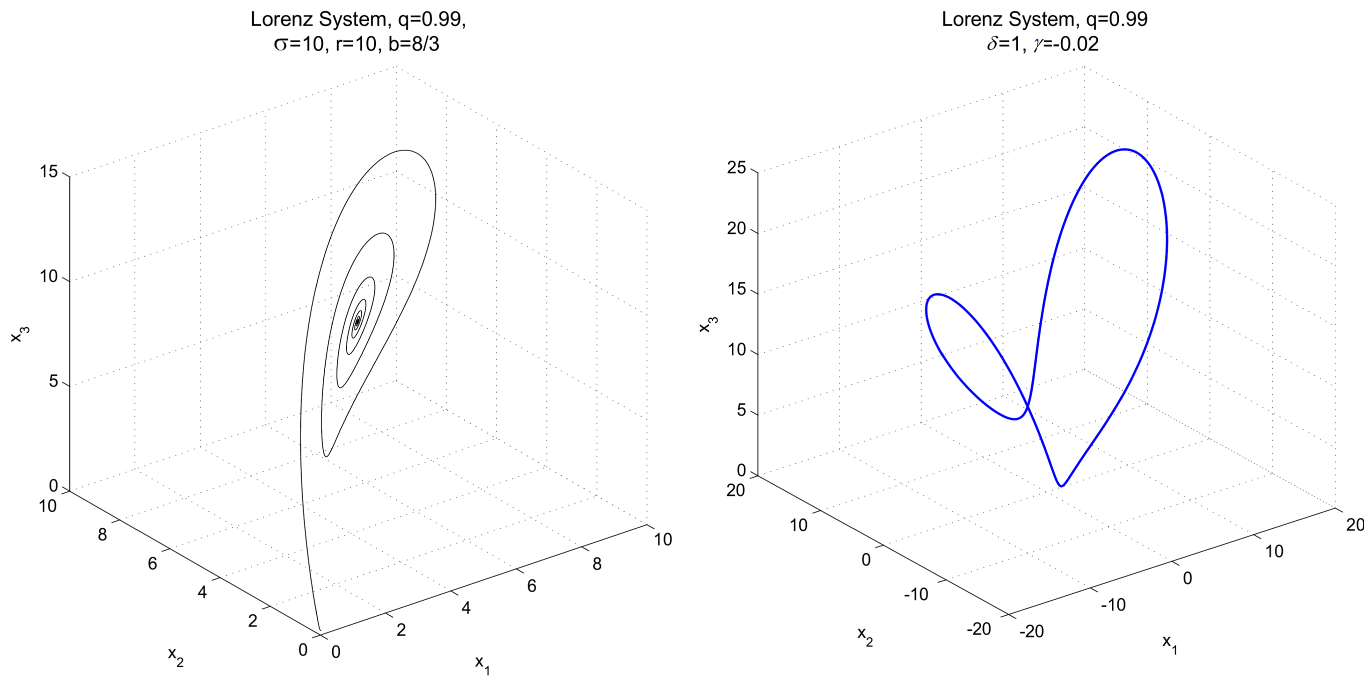


Fig. 10. Lorenz system for $(\sigma, r, b) = (10, 10, 8/3)$ and $q = 0.99$; (a) attractive point; (b) chaotic attractor obtained by PP method with $\delta = 1$ and $\gamma = -0.02$.



(a) (b)

Fig. 11. Lorenz system for $(\sigma, r, b) = (10, 220, 8/3)$ and $q = 0.99$; (a) stable cycle (without PP method); (b) chaotic attractor obtained by PP method with $\delta = 1$ and $\gamma = -0.01$.



(a) (b)

Fig. 12. Lorenz system for $(\sigma, r, b) = (10, 10, 10)$ and $q = 0.99$; (a) attractive point (without PP method); (b) stable cycle obtained by PP method with $\delta = 1$ and $\gamma = -0.02$.